Proceedings of the Jangjeon Mathematical Society
22 (2019), No. 3. pp. 365 – 368
www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2019.22.3.365

FINITE-DIMENSIONAL FINALLY PRECONTINUOUS UNITARY PSEUDOREPRESENTATIONS OF ALMOST CONNECTED LOCALLY COMPACT GROUPS

A. I. Shtern

Abstract. We describe the structure of finally precontinuous finite-dimensional unitary pseudorepresentations of almost connected locally compact groups.

§ 1. Introduction

The structure of unitary finite-dimensional pseudorepresentations of connected Lie groups is known [1]. In the present paper, using results of [1–4], we discuss the structure of unitary finite-dimensional pseudorepresentations of almost connected locally compact groups.

§ 2. Preliminaries

The finite-dimensional quasirepresentations of groups have the general structure described in Theorem 2.5.13 of [4]; see also [1–3]. This theorem shows that a unitary pseudorepresentation is a component of the generic locally bounded pseudorepresentation, and therefore the structure of these objects is of importance.

Recall Dong Hoon Lee's supplement theorem (Theorem 2.13 of [5]): every almost connected locally compact group G with the connected component G_0 (i.e., a locally compact group G for which the quotient group G/G_0

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25, 22A10.
Key words and phrases. Almost connected locally compact group, pseudorepresentation, unitary pseudorepresentation.

is compact) admits a totally disconnected compact subgroup D such that $G = G_0 D$.

We also recall the following notion.

Lemma 1. *Let G be an almost connected locally compact group, and let $\mathcal{N}$ be the family of compact normal subgroups $N \neq \{e\}$ such that G/N is a (not necessarily connected) Lie group. Then $\mathcal{N}$ is a nontrivial filter basis convergent to $\{e\}$.*

Proof. This follows immediately from the Gleason–Montgomery–Zippin–Yamabe theorem.

Recall that a finite-dimensional locally bounded (in particular, unitary) representation of G is said to be *finally precontinuous* if the set $\mathrm{FDG}(\pi) = \bigcap_{N\in\mathcal{N}} \overline{\pi(N)}$ is the singleton formed by the identity operator on the representation space of π [4]. As is well known, this condition implies that the restriction of π to the commutator subgroup G' of G is continuous with respect to the intrinsic Lie topology of the Lie group G' [6].

§ 3. Main results

The above results make it possible to give a short proof of the following theorem.

Theorem 2. *Let G be an almost connected locally compact group, let G_0 be the connected component of G, let R be the radical of G_0, let L be a Levi subgroup of G_0, and let D be a totally disconnected compact subgroup such that $G = G_0 D$. Let π be a unitary finally precontinuous locally bounded pseudorepresentation of G in a finite-dimensional complex vector space E with a sufficiently small defect. Then there is a compact normal subgroup N of G such that the pseudorepresentation π can be regarded as a pseudorepresentation ρ of the Lie group G/N. The connected component $(G/N)_0$ has finite index in G/N. If the group $(G/N)_0$ has a nontrivial Hermitian symmetric quotient group, then the pseudorepresentation ρ is the direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G, some one-dimensional Guichardet–Wigner pseudorepresentations (i.e., mappings of the form $g \to \exp(ir\theta(g))$, $g \in G$, for some $r \in \mathbb{R}$, where θ stands for a Guichardet–Wigner pseudocharacter on G, see [4]), and some G-central unitary characters χ of the group R (i.e., $\varphi(k) = \varphi(gkg^{-1})$ for all $k \in R$ and $g \in G$). If the group G has no nontrivial Hermitian symmetric quotient groups, then the pseudorepresentation*

ρ is the direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G and some G-central unitary characters of the group R.

Proof. It follows from [6] (see [7]) for preliminaries) that there is a compact normal subgroup N of G such that the unitary pseudorepresentation π can be regarded as a unitary pseudorepresentation ρ of the Lie group G/N. By Theorem 3.3.17 of [4], part 2, if the group G has a nontrivial Hermitian symmetric quotient group, then ρ is a direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G, some one-dimensional Guichardet–Wigner pseudorepresentations (i.e., mappings of the form

$$g \to \exp(i\chi(g)), \qquad g \in G,$$

where χ stands for a Guichardet–Wigner pseudocharacter on G [4]), and some G-central unitary characters of the group R and, if the group G has no nontrivial Hermitian symmetric quotient groups, then ρ is the direct sum of (ordinary) products of some continuous irreducible unitary representations of the maximal compact quotient group of G and some G-central unitary characters of the group R. This completes the proof of the theorem.

§ 4. Discussion

Theorem 2 motivates the following conjecture. Recall that a pseudorepresentation is said to be pure if its restriction to every commutative subgroup is an ordinary representation of the subgroup.

Conjecture. *Every finite-dimensional unitary finally precontinuous pseudorepresentation of an almost connected locally compact group is pure.*

Since, by Theorem 2, the restriction of the pseudorepresentation ρ (constructed for such a pseudorepresentation π) to the connected component $(G/N)_0$ of the identity element e of the group G/N is expressed in terms of ordinary representations and pseudocharacters, which are automatically pure, it follows that the restriction of π to G_0 is pure. Since the group $(G/N)/(G/N)_0$ is finite, it follows that the corresponding group D is finite, and hence the restriction of ρ to D is an ordinary representation of D. This gives hope that the conjecture is correct.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

Funding

The research was partially supported by the Scientific Research Institute of System Analysis, Russian Academy of Sciences (FGU FNTs NIISI RAN), theme NIR 0065-2019-0007.

References

1. A. I. Shtern, *Irreducible locally bounded finite-dimensional pseudorepresentations of connected locally compact groups*, Russ. J. Math. Phys. **25** (2018 issue2), 239–240.
2. A. I. Shtern, *Locally bounded finally precontinuous finite-dimensional quasirepresentations of connected locally compact groups*, Mat. Sb. **208** (2017), no. 10, 149–170; English transl., Sb. Math. **208** (2017), no. 10, 1557–1576.
3. A. I. Shtern, *Continuity Conditions for Finite-Dimensional Locally Bounded Representations of Connected Locally Compact Groups*, Russ. J. Math. Phys. **25** (2018), no. 5, 345–382.
4. A. I. Shtern, *Finite-Dimensional Quasirepresentations of Connected Lie Groups and Mishchenko's Conjecture*, Fundam. Prikl. Mat. **13** (2007), no. 7, 85–225; J. Math. Sci. (N.Y.) **159** (2009), no. 5, 653–751.
5. D. H. Lee, *Supplements for the identity component in locally compact groups*, Math. Z. **104** (1968), no. 1, 28–49.
6. A. I. Shtern, *Continuity conditions for locally bounded finite-dimensional representations of almost connected locally compact groups*, Proc. Jangjeon Math. Soc. **21** (2018), no. 4, 779–782.
7. A. I. Shtern, *Continuity conditions for finite-dimensional locally bounded representations of connected locally compact groups*, Russ. J. Math. Phys. **25** (2018), no. 3, 345–382.

Department of Mechanics and Mathematics,
Moscow State University,
Moscow, 119991 Russia, and
Scientific Research Institute of System Analysis (FGU FNTs NIISI RAN),
Russian Academy of Sciences,
Moscow, 117312 Russia
E-mail: aishtern@mtu-net.ru, rroww@mail.ru